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# A Renormalization Result for the Intersection Local Time of Lattice Random Walks in $d \geqslant 3$ Dimensions

Sergio Albeverio[1] and Xian Yin Zhou[1,2]

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In this paper we derive some general conditions on stable walks in $Z^d$, under which the central limit theorem holds for their normalized intersection local time. In particular, we prove that the process given by the normalized intersection local time of the simple random walk in $Z^d$, with $d \geqslant 3$, is weakly convergent to the standard Brownian motion.

**KEY WORDS:** Renormalization; random walk; Brownian motion; central limit theorem; intersection local time.

## 1. INTRODUCTION

Let $(\xi_i,\ i \geqslant 1)$ be independent, identically distributed random variables in $Z^d$ on a probability space $(\Omega, \mathscr{F}, P)$. Let $\{X_n\}_{n \geqslant 0}$ be the random walk in $Z^d$ defined by

$$X_n = \sum_{i=1}^{n} \xi_i, \qquad \forall n \geqslant 1$$

and $\{P_x\}$ be the probability law of $\{X_n\}_{n \geqslant 0}$. As in ref. 6, we introduce the following two assumptions on the random walk $\{X_n\}_{n \geqslant 0}$. Let $\beta \in (0, 2]$ be a given constant.

**Assumption $A_1$.** There exists a function $b(n)$ of regular variation of index $1/\beta$ such that

$$b^{-1}(n)\, X_n \xrightarrow{(d)} U_1, \qquad n \to \infty$$

[1] Fakultät für Mathematik, Ruhr-Universität Bochum, D-44780 Bochum 1, Germany; BiBoS; SFB 237 Bochum-Essen-Düsseldorf; CERFIM, Locarno.

[2] Department of Mathematics, Beijing Normal University, Beijing 100875, China.



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where $U_1$ is a nondegenerate stable random variable of index $\beta$ in $R^d$, and $\xrightarrow{(d)}$ means that the convergence holds in distribution.

**Assumption A$_2$.** With $P_0$-probability one, $\{X_n\}_{n\geqslant 0}$ does not stay on a proper subgroup of $Z^d$.

As explained in ref. 7, one may assume that the function $b$ is continuous and monotonically increasing from $R_+$ onto $R_+$, and that $b(0)=0$.

Let $\{B_t\}_{t\geqslant 0}$ be the Brownian motion in $R^d$ on a probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \mu)$, and $B_0=0$. Let

$$\alpha(x, A)=\iint_A \delta_x(B_t-B_s)\,dt\,ds, \qquad \forall A\subset R_+^2$$

where $\delta_x$ is the $\delta$-function. There have been many results about the existence of $\alpha(x, A)$ (e.g., see ref. 4 and references therein). In general, $\alpha(x, A)$ is called the intersection local time of $\{B_t\}$ on the set $A$.

Let

$$I(n)=\sum_{i=1}^{n}\sum_{j=i+1}^{n}\delta(X_i, X_j)$$

Corresponding to the definition of $\alpha(x, A)$, the random variable $I(n)$ is called the intersection local time of the random walk $\{X_n\}_{n\geqslant 0}$ (or briefly, the discrete intersection local time) in this paper.

For $d=2$, there have been many investigations about the renormalization for the intersection local time of Brownian motion and random walks (e.g., see refs. 4, 6, 8, and 10). For $d=3$, Yor[14] obtained a renormalization result for the intersection local time of Brownian motion by means of some tools in stochastic analysis and some representations for the intersection local time $\alpha(x, A)$.[9,13] More precisely, Yor proved that[14]

$$\begin{aligned}&(B_t, (\log\varepsilon^{-1})^{-1/2}[2\pi\alpha(0, T_t^\varepsilon)-t\varepsilon^{-1/2}E\,|B_1|^{-1}], t\geqslant 0)\\ &\xrightarrow{(d)} (B_t, 2^{1/2}\beta_t, t\geqslant 0), \qquad \varepsilon\to 0 \qquad (1.1)\end{aligned}$$

where $\{\beta_t\}_{t\geqslant 0}$ is a standard Brownian motion which is independent of $\{B_t\}$, and

$$T_t^\varepsilon=\{(x, y)\colon 0\leqslant x\leqslant y-\varepsilon;\, x, y\leqslant t\}$$

If $\{X_n\}$ is the simple random walk in $Z^3$, one can also obtain a renormalization result for $I(n)$ by using Yor's result as above [i.e., (1.1)] and some estimate (e.g., ref. 1) on the convergence rate of the invariance principle for the intersection local time (see ref. 1). However, it is difficult to get a renormalization result for the intersection local time of some stable random walks and higher-dimensional simple random walks by using the above approach.

The main purpose of this paper is to derive a renormalization result for the intersection local time of random walks in $Z^d$ by using another approach. This approach is based on Le Gall and Rosen,[7] where many limit theorems are obtained for the range of stable random walks in $Z^d$. In Section 2, we first derive some general conditions on the random walk $\{X_n\}$ such that the normalized intersection local time can converge weakly to the standard normal distribution under these conditions (see Theorem 2.3 below). In Section 3, we check that the simple random walk in $Z^d$ with $d \geqslant 3$ satisfies the general conditions derived in Section 2, and prove an analogous result to (1.1) for the normalized intersection local time of the simple random walk in $Z^d$ with $d \geqslant 3$.

At the end of this section, we remark that there is a connection between the construction of a polymer measure and the renormalization result for the intersection local time. In fact, one can use the renormalization result for the intersection local time of Brownian motion (or simple random walk) to construct the polymer measure in two dimensions.[1,6,10,11] For $d=3$, (1.1) (or Theorem 3.4 below) can help to understand the renormalization result for the normalization constant given in the definition of the polymer measure (see ref. 1, Theorem 1.1, ref. 3, §3, or ref. 12). We hope our results (see Theorem 3.4 below) can also help to understand the renormalization result for the normalization constant given in the definition of the Domb–Joyce model for $d \geqslant 4$ (for the definition, see ref. 5).

## 2. CENTRAL LIMIT THEOREM

The main aim of this section is to derive a general condition under which the central limit theorem holds for the normalized intersection local time of random walks in $Z^d$. In this section we continue to use the notation introduced in Section 1, and assume that all random walks here satisfy the assumptions $A_1$ and $A_2$. Let

$$P_n(x, y) = P_x(X_n = y) = P_0(X_n = x - y)$$

For convenience, we first recall some estimates on the transition probability $P_n(x, y)$ (see ref. 7, Proposition 2.4). Let $\tau$ be the period of $\{X_n\}$. Then there is a constant $C_1 \in (0, \infty)$ such that:

(i) $P_n(0,0)=0$ if $n$ is not a multiple of $\tau$.

(ii) $\lim_{n\to\infty} b^d(n\tau)\, P_{n\tau}(0,0)=\tau p_1(0)$.

(iii) $P_n(0,x)\leqslant C_1 b(n)^{-d}$.

Let $\{X_1(n)\}_{n\geqslant 0}$ and $\{X_2(n)\}_{n\geqslant 0}$ be independent random walks on $(\Omega, \mathscr{F}, P)$ with the same distribution as $\{X_n\}$ introduced in Section 1, and $X_1(0)=X_2(0)=0$. Let

$$\beta(n)=\sum_{i=0}^{n}\sum_{j=0}^{n} P(X_1(i)=X_2(j))$$

Then we have the following result.

**Lemma 2.1.** There is a constant $C_2\in(0,\infty)$ such that

$$E\left(\sum_{i=0}^{n}\sum_{j=0}^{n}\delta(X_1(i),X_2(j))\right)^4\leqslant C_2\beta^4(n), \qquad \forall n\geqslant 1$$

where $E$ is the expectation with respect to $P$.

Before proving this lemma, let us first recall an elementary property of $b(n)$ [see ref. 7, (2.a)]:

(iv) The following holds for any given $N\in(0,\infty)$:

$$b(Nn)\leqslant O(b(n)), \qquad \forall n\geqslant 1$$

*Proof of Lemma 2.1.* For convenience, in the following proof we assume that $b(n)$ is positive for any $n\geqslant 0$. It is easy to show that

$$\begin{aligned}
&E\left(\sum_{i=0}^{n}\sum_{j=0}^{n}\delta(X_1(i),X_2(j))\right)^4\\
&=4!\,E\left(\sum_{0\leqslant i_1\leqslant\cdots\leqslant i_4\leqslant n}\ \sum_{0\leqslant j_1,\ldots,j_4\leqslant n}\ \prod_{k=1}^{4}\delta(X_1(i_k),X_2(j_k))\right)\\
&=4!\sum_{0\leqslant i_1\leqslant\cdots\leqslant i_4\leqslant n}\ \sum_{0\leqslant j_1,\ldots,j_4\leqslant n} E(P_{i_1}(0,X_2(j_1))\,P_{i_2-i_1}(0,X_2(j_2)-X_2(j_1))\\
&\quad\times P_{i_3-i_2}(0,X_2(j_3)-X_2(j_2))\,P_{i_4-i_3}(0,X_2(j_4)-X_2(j_3)))
\end{aligned}$$

Let $\{\tau_1(j),\tau_2(j),\tau_3(j),\tau_4(j)\}=\{j_1,j_2,j_3,j_4\}$ satisfy

$$\tau_1(j)\leqslant\tau_2(j)\leqslant\tau_3(j)\leqslant\tau_4(j)$$

Let

$$f_k(x) = P_{\tau_{k+1}(j)-\tau_k(j)}(0, x), \qquad k = 1, 2, 3$$
$$g_k(x) = P_{i_{k+1}-i_k}(0, x), \qquad k = 1, 2, 3$$

and

$$\mathscr{D} = \sum_{x_1, x_2, x_3 \in Z^d} \prod_{i=1}^{3} (f_i(x_i)\, g_i(v_i(x)))$$

where $v_i(x) = \sum_{j=1}^{3} a_{ij} x_j$, $a_{ij} \in \{-1, 0, 1\}$, $i, j = 1, 2, 3$, and the matrix $(a_{ij})_{1 \leqslant i,j \leqslant 3}$ is not singular. Note that $X_2(\tau_1(j))$ is independent of $\{X_2(j_l) - X_2(j_k),\ 1 \leqslant l, k \leqslant 4\}$. By property (iii) we can show that

$$E\left(\sum_{i=0}^{n} \sum_{j=0}^{n} \delta(X_1(i), X_2(j))\right)^4$$
$$\leqslant 4! \sum_{0 \leqslant i_1 \leqslant \cdots \leqslant i_4 \leqslant n} \sum_{0 \leqslant j_1, \ldots, j_4 \leqslant n} \max_{z \in Z^d} P_{i_1 - \tau_1(j)}(0, z)\, \mathscr{D}$$
$$\leqslant \sum_{0 \leqslant i_1 \leqslant \cdots \leqslant i_4 \leqslant n} \sum_{0 \leqslant \tau_1(j) \leqslant \cdots \leqslant \tau_4(j) \leqslant n} O(b^{-d}(i_1 + \tau_1(j)))\, \mathscr{D}$$

By property (ii) and the monotonicity property of the function $b$ we can show that

$$\beta(n) = O\left(\sum_{i=0}^{n} \sum_{j=0}^{n} b^{-d}(i+j)\right)$$

Thus, to prove the desired result, we only need to show that for any given $(a_{ij})$ with the properties mentioned as above

$$\mathscr{D} \leqslant O(1) \sum_{m=1}^{3} \left( \prod_{k=1}^{3} \prod_{l=m}^{m+2} b^{-d}(i_{k+1} - i_k + \tau_{l+1}(j) - \tau_l(j)) \right.$$
$$+ \prod_{k=1}^{3} \prod_{l=m}^{m+2} b^{-d}(i_{k+1} - i_k + \tau_{l+2}(j) - \tau_{l+1}(j))$$
$$\left. + \prod_{k=1}^{3} \prod_{l=m}^{m+2} b^{-d}(i_{k+1} - i_k + \tau_{l+3}(j) + \tau_{l+2}(j)) \right)$$

where $\tau_{l+1}(j) - \tau_l(j) = \tau_{k+1}(j) - \tau_k(j)$ if $l = 3m + k + 1$ for some $k \in \{1, 2, 3\}$.

To save space, we will prove the above estimate only for the following case:

$$v_1(x) = x_1 + x_2 + x_3, \qquad v_2(x) = x_2 + x_3, \qquad v_3(x) = x_1 + x_3$$

In fact, by a similar argument one can also prove the desired estimate for the other cases. In the present case, we have for $\tau_2(j)-\tau_1(j)\leqslant i_2-i_1$:

$$\begin{aligned}\mathscr{D} &\leqslant O(b^{-d}(i_2-i_1)) \sum_{x_1\in Z^d} P_{\tau_2(j)-\tau_1(j)}(0,x_1) \sum_{x_3\in Z^d} P_{\tau_4(j)-\tau_3(j)}(0,x_3)\\ &\quad\times P_{i_4-i_3}(0,x_1+x_3) \sum_{x_2\in Z^d} P_{\tau_3(j)-\tau_2(j)}(0,x_2)\, P_{i_3-i_2}(0,x_2+x_3)\\ &\leqslant O(b^{-d}(i_2-i_1)\, b^{-d}(\tau_3(j)-\tau_2(j)+i_3-i_2)\, b^{-d}(\tau_4(j)-\tau_3(j)+i_4-i_3))\\ &\leqslant O(b^{-d}(\tau_2(j)-\tau_1(j)+i_2-i_1)\, b^{-d}(\tau_3(j)-\tau_2(j)+i_3-i_2)\\ &\quad\times b^{-d}(\tau_4(j)-\tau_3(j)+i_4-i_3))\end{aligned}$$

by properties (iii) and (iv). Note that

$$\begin{aligned}&\sum_{x_2\in Z^d} P_{\tau_3(j)-\tau_2(j)}(0,x_2) \sum_{x_1\in Z^d} P_{i_2-i_1}(0,x_1+x_2)\, P_{i_4-i_3}(0,x_1)\\ &\qquad\leqslant \begin{cases} O(b(\tau_3(j)-\tau_2(j)) & \text{if } \tau_3(j)-\tau_2(j)\geqslant i_4-i_3\\ O(b(i_4-i_3)) & \text{if } i_4-i_3\geqslant \tau_3(j)-\tau_2(j)\end{cases}\end{aligned}$$

By property (iv) again we know that the right-hand sides of the above estimate are less than or equal to

$$O(b^{-d}(\tau_3(j)-\tau_2(j)+i_4-i_3))$$

Thus, we have for $\tau_2(j)-\tau_1(j)>i_2-i_1$

$$\begin{aligned}\mathscr{D} &\leqslant O(b^{-d}(\tau_2(j)-\tau_1(j)+i_2-i_1)) \sum_{x_2\in Z^d} P_{\tau_3(j)-\tau_2(j)}(0,x_2)\\ &\quad\times \sum_{x_3} P_{\tau_4(j)-\tau_3(j)}(0,x_3)\, P_{i_3-i_2}(0,x_2+x_3)\\ &\quad\times \sum_{x_1} P_{i_2+i_1}(0,x_1+x_2)\, P_{i_4-i_3}(0,x_1)\\ &\leqslant O(b^{-d}(\tau_2(j)-\tau_1(j)+i_2-i_1)\, b^{-d}(\tau_4(j)-\tau_3(j)+i_3-i_2)\\ &\quad\times b^{-d}(\tau_3(j)-\tau_2(j)+i_4-i_3)) \quad \blacksquare\end{aligned}$$

This proves the desired result.

Let

$$\alpha(n)=\beta^{-2}(n)\, E_0(I(n)-E_0I(n))^2$$

where $E_x$ is the expectation with respect to $P_x$.

**Lemma 2.2.** Assume that the sequence $\{\alpha^{-1/2}(2^{n+1})\}_{n\geqslant 0}$ is bounded. If there are sequences $\{\tilde{\alpha}(n)\}_{n\geqslant 1}$ and $\{\tilde{\beta}(n)\}_{n\geqslant 1}$ and constants $N_0<\infty$ and $\varepsilon\in(0,1)$ such that

$$\frac{2\tilde{\alpha}^2(2^n)\,\tilde{\beta}^4(2^n)}{\tilde{\alpha}^2(2^{n+1})\,\tilde{\beta}^4(2^{n+1})}\leqslant 1-\varepsilon, \qquad \forall n\geqslant 0 \tag{2.1}$$

and

$$\sup_{n\geqslant 1}\left(\frac{\alpha(n)}{\tilde{\alpha}(n)}\vee\frac{\tilde{\alpha}(n)}{\alpha(n)}\vee\frac{\beta(n)}{\tilde{\beta}(n)}\vee\frac{\tilde{\beta}(n)}{\beta(n)}\right)\leqslant N_0 \tag{2.2}$$

then there is a constant $C_3\in(0,\infty)$ satisfying

$$E_0\,|I(2^n)-E_0I(2^n)|^4\leqslant C_3\alpha^2(2^n)\,\beta^4(2^n), \qquad \forall n\geqslant 1$$

In addition, if

$$\sum_{k=1}^{n}\alpha^{1/2}(2^k)\,\beta(2^k)\leqslant O(\min_{2^n\leqslant k\leqslant 2^{n+1}}(\alpha^{1/2}(k)\,\beta(k))), \qquad \forall n\geqslant 1 \tag{2.3}$$

then we have

$$E_0\,|I(n)-E_0I(n)|^4\leqslant C_3\alpha^2(n)\,\beta^4(n), \qquad \forall n\geqslant 1$$

under the same conditions as before [i.e. (2.1) and (2.2)].

*Proof.* The following idea is basically from the proof of ref. 7, Theorem 4.5. For $k\geqslant 0$, let

$$J_k(n)=\sum_{i=kn+1}^{(k+1)n}\sum_{j=i+1}^{(k+1)n}\delta(X_i,X_j)$$

and

$$R_k(n)=J_k(2^n)-E_0J_k(2^n)$$

Then,

$$R_0(n+1)=R_0(n)+R_1(n)+\sum_{i=1}^{2^n}\sum_{j=2^n+1}^{2^{n+1}}(\delta(X_i,X_j)-E_0\delta(X_i,X_j))$$

Note that

$$E_0\,|R_0(n)+R_1(n)|^4\leqslant 2E_0\,|R_0(n)|^4+C_4\beta^4(2^n)\,\alpha^2(2^n)$$

for some constant $C_4 \in (0, \infty)$. By Lemma 2.1, there is a constant $C_5 \in (0, \infty)$ such that

$$\begin{aligned}(E_0\,|R_0(n+1)|^4)^{1/4} &\leqslant (E_0\,|R_0(n)+R_1(n)|^4)^{1/4} \\ &\quad + \left\{E_0\left(\sum_{i=1}^{2^n}\sum_{j=2^n+1}^{2^{n+1}}[\delta(X_i,X_j)-E_0\delta(X_i,X_j)]\right)^4\right\}^{1/4} \\ &\leqslant [2E_0\,|R_0(n)|^4 + C_4\beta^4(2^n)\,\alpha^2(2^n)]^{1/4} + C_5\beta(2^n)\end{aligned}$$

Hence, we have for some constants $C_4'$ and $C_5' \in (0, \infty)$

$$\begin{aligned}&[\tilde{\alpha}^{-2}(2^{n+1})\,\tilde{\beta}^{-4}(2^{n+1})\,E_0\,|R_0(n+1)|^4]^{1/4} \\ &\quad\leqslant \left(\frac{2\tilde{\alpha}^2(2^n)\,\tilde{\beta}^4(2^n)}{\tilde{\alpha}^2(2^{n+1})\,\tilde{\beta}^4(2^{n+1})}\,\tilde{\alpha}^{-2}(2^n)\,\tilde{\beta}^{-4}(2^n)\,E_0\,|R_0(n)|^4 + C_4'\right)^{1/4} \\ &\quad\quad + C_5'\alpha^{-1/2}(2^{n+1})\end{aligned}$$

By our hypothesis on $\{\alpha(2^n)\}$ and (2.1) we can show that there is a constant $C_6 \in (0, \infty)$ such that

$$\begin{aligned}&[\tilde{\alpha}^{-2}(2^{n+1})\,\tilde{\beta}^{-4}(2^{n+1})\,E_0\,|R_0(n+1)|^4]^{1/4} \\ &\quad\leqslant [(1-\varepsilon)\,\tilde{\alpha}^{-2}(2^n)\,\tilde{\beta}^{-4}(2^n)\,E_0\,|R_0(n)|^4 + C_6]^{1/4}\end{aligned}$$

Thus, by induction one can easily show that

$$E_0\,|R_0(n+1)|^4 \leqslant C_7\tilde{\alpha}^2(2^{n+1})\,\tilde{\beta}^4(2^{n+1}), \qquad \forall n \geqslant 1$$

for some constant $C_7 \in (0, \infty)$.

Now we assume (2.3) holds. For any $k = 1,..., 2^n - 1$, there are $l_0,..., l_{n-1} \in \{0, 1\}$ such that

$$k = \sum_{i=0}^{n-1} l_i 2^i$$

Then, from the argument given before we can see that

$$\begin{aligned}&[E_0\,|I(2^n+k) - E_0 I(2^n+k)|^4]^{1/4} \\ &\quad\leqslant O(\alpha^{1/2}(2^n)\,\beta(2^n)) + [E_0\,|I(k) - E_0 I(k)|^4]^{1/4}\end{aligned}$$

By induction we can show that

$$[E_0\,|I(k) - E_0 I(k)|^4]^{1/4} \leqslant \sum_{i=0}^{n-1} l_i O(\alpha^{1/2}(2^i)\,\beta(2^i))$$

In fact by our hypothesis (2.3) we know that

$$[E_0(I(2^n+k)-E_0(2^n+k))^4]^{1/4} \leqslant O(\alpha^{1/2}(2^n+k)\,\beta(2^n+k))$$

and this implies the desired result, completing the proof of Lemma 2.2. ∎

The main result in this section is as follows.

**Theorem 2.3.** Assume that (2.1) and (2.2) are satisfied, and the sequence $\{\alpha^{-1/2}(2^{n+1})\}_{n\geqslant 0}$ is bounded. If there is a sequence $\{m_n\}_{n\geqslant 0}$ such that $m_n \in [1, n]$, $\forall n \geqslant 1$, $\lim_{n\to\infty} m_n = \infty$, and

$$\lim_{n\to\infty} \frac{\sum_{l=1}^{m_n} 2^{l/2}\beta(2^{n-l})}{\beta(2^{n-m_n})\,2^{m_n/2}\alpha^{1/2}(2^{n-m_n})} = 0 \tag{2.4}$$

then the following holds:

$$\frac{I(2^n)-E_0 I(2^n)}{\beta(2^{n-m_n})\,2^{m_n/2}\alpha^{1/2}(2^{n-m_n})} \xrightarrow{(d)} N(0,1), \qquad n\to\infty$$

where $N(0, 1)$ is a random variable with the standard normal distribution.

*Proof.* Let

$$T(l,k,n) = [(2k-2)\,2^{n-l}, (2k-1)2^{n-l}] \times [(2k-1)2^{n-l}+1, (2k)2^{n-l}]$$

$$\Delta_m(n) = \bigcup_{l=1}^{m} \bigcup_{k=1}^{2^{l-1}} T(l,k,n)$$

and

$$J(m,n) = \sum_{(i,j)\in\Delta_m(n)} \delta(X_i, X_j)$$

Then, we have

$$I(2^n) = \sum_{i=0}^{2^{m_n}-1} J_i(2^{n-m_n}) + J(m_n, n)$$

where

$$J_0(2^{n-m_n}),\dots, J_{2^{m_n}-1}(2^{n-m_n})$$

were defined in the proof of Lemma 2.2. It is clear that the random variables

$$J_0(2^{n-m_n}),\dots, J_{2^{m_n}-1}(2^{n-m_n})$$

are independent identically distributed, and there is a constant $C_8 \in (0, \infty)$ such that

$$(E_0 \, |J(m_n, n) - E_0 J(m_n, n)|^2)^{1/2}$$
$$\leqslant \sum_{l=1}^{m_n} 2^{(l-1)/2} \operatorname{Var}\left( \sum_{(i,j) \in T(l,1,n)} \delta(X_i, X_j) \right)^{1/2}$$
$$\leqslant \sum_{l=1}^{m_n} 2^{(l-1)/2} \left( E \left( \sum_{i=0}^{2^{n-l}} \sum_{j=0}^{2^{n-l}} \delta(X_1(i), X_2(j)) \right)^2 \right)^{1/2}$$
$$\leqslant C_8 \sum_{l=1}^{m_n} 2^{l/2} \beta(2^{n-l}) \tag{2.5}$$

by Lemma 2.1. Thus, the desired result follows from the following convergence:

$$\frac{\sum_{i=0}^{2^{m_n}-1} (J_i(2^{n-m_n}) - E_0 J_i(2^{n-m_n}))}{\beta(2^{n-m_n}) \, 2^{m_n/2} \alpha^{1/2}(2^{n-m_n})} \xrightarrow{(d)} N(0, 1) \tag{2.6}$$

We now prove (2.6). Note that [we denote $E_0(J - E_0 J)^2$ by $\operatorname{Var}(J)$]

$$E_0 \left( \sum_{i=0}^{2^{m_n}-1} [J_i(2^{n-m_n}) - E_0 J_i(2^{n-m_n})] \right)^2$$
$$= \sum_{i=0}^{2^{m_n}-1} \operatorname{Var}(J_i(2^{n-m_n}))$$
$$= 2^{m_n} \beta^2(2^{n-m_n}) \, \alpha(2^{n-m_n})$$

Thus, to prove (2.6) we only need to prove that the Lindeberg condition is satisfied,[2] i.e.

$$\lim_{n \to \infty} \left\{ \left[ \sum_{i=0}^{2^{m_n}-1} \int_{|x| \geqslant \varepsilon 2^{m_n/2} \beta(2^{n-m_n}) \alpha^{1/2}(2^{n-m_n})} x^2 \, dF_i(x) \right] \Big/ [2^{m_n} \beta^2(2^{n-m_n}) \, \alpha(2^{n-m_n})]^{-1} \right\} = 0, \qquad \forall \varepsilon > 0 \tag{2.7}$$

where $F_i(x)$ is the distribution function of $J_i(2^{n-m_n}) - E_0 J_i(2^{n-m_n})$. Indeed, by Lemma 2.2 we know that

$$\sum_{i=0}^{2^{m_n}-1} E_0 \, |J_i(2^{n-m_n}) - E_0 J_i(2^{n-m_n})|^4$$
$$\leqslant C_3 2^{m_n} \alpha^2(2^{n-m_n}) \, \beta^4(2^{n-m_n}) \tag{2.8}$$

Using this, we can see that Lyapunov's condition is satisfied with $\delta = 2$ (see ref. 2, 27.16). Therefore the Lindeberg condition (2.7) is satisfied (see ref. 2, p. 312). Thus, (2.6) is proved, which completes the proof of Theorem 2.3.

## 3. WIENER PROCESS BEHAVIOR

In Section 2 we derived a general condition under which the central limit theorem holds for the normalized intersection local time of some lattice random walks. The main aim of this section is to show that some random walks in $Z^d$ satisfy all assumptions given in Sections 1 and 2 if $d \geqslant 3$. Since there is no nice estimate [as in (a)–(c) below] for the transition probabilities of a general stable random walk, we are unable to get a desirable estimate for the variance of the random variable $I(n)$ [i.e., $\mathrm{Var}(I(n))$] as in Lemma 3.1 below. Nevertheless, our approach given in the proof of Theorem 3.4 below is in principle suitable for some stable random walks with some symmetric properties. For simplicity, in this section we restrict our attention to the case of simple random walks. We always assume that $\{X_n\}$ is the simple random walk in $Z^d$. We first recall some results on the transition probability of the simple random walk in $Z^d$. Let

$$\bar{P}_n(x) = 2\left(\frac{d}{2\pi n}\right)^{d/2} \exp\left(-\frac{d\,|x|^2}{2n}\right)$$

Then we have the following properties.

(a) There is a constant $C_1 \in (0, \infty)$ such that (see ref. 5, Theorem 1.2.1)

$$|P_n(0, x) - \bar{P}_n(x)| \leqslant C_1 n^{-(d+2)/2}$$

$$|P_n(0, x) - \bar{P}_n(x)| \leqslant C_1 |x|^{-2} n^{-d/2}$$

if $P_n(0, x) > 0$.

(b) There is a constant $C_2 \in (0, \infty)$ for any given $\alpha \in (1/2, 2/3)$ such that (see ref. 5, Proposition 1.2.5)

$$|P_n(0, x) - \bar{P}_n(x)| \leqslant C_2 n^{3\alpha - 2} \bar{P}_n(x)$$

if $|x| \leqslant n^{\alpha}$ and $P_n(0, x) > 0$.

(c) There is a constant $C_3 \in (0, \infty)$ such that for $|x| > n$ [see ref. 1, §4(i)]

$$P_n(0, x) \leqslant C_3 \bar{P}_{nd}(x)$$

**Lemma 3.1.** (i) For $d=3$, the following holds:

$$\operatorname{Var}(I(n))=O(n\log n)$$

(ii) For $d\geqslant 4$, there is a constant $k_d\in(0,\infty)$ such that

$$\lim_{n\to\infty}\frac{\operatorname{Var}(I(n))}{n}=k_d$$

*Proof.* (i) The following proof is adapted from the proof of ref. 4, Proposition 6.4.1. Let

$$q(i_1,j_1;i_2,j_2)=P_0(X(i_1)=X(j_1),\,X(i_2)=X(j_2))-p_{j_1-i_1}p_{j_2-i_2}$$

where $p_n=P_n(0,0)$. Let

$$L_1=\sum_{1\leqslant i_1\leqslant i_2<j_2\leqslant j_1\leqslant n}q(i_1,j_1;i_2,j_2)$$

$$L_2=\sum_{1\leqslant i_1\leqslant i_2\leqslant j_1<j_2\leqslant n}q(i_1,j_1;i_2,j_2)$$

It is clear that

$$L_1\leqslant\operatorname{Var}(I(n))\leqslant L_1+L_2$$

As in the proof of ref. 5, Proposition 6.4.1, by using the Markov property and property (a), one can show that for some constants $C_4$, $C_5\in(0,\infty)$

$$\begin{aligned}
L_1&=\sum_{1\leqslant i_1\leqslant i_2<j_2\leqslant j_1\leqslant n}p_{j_2-i_2}(p_{j_1-i_1-(j_2-i_2)}-p_{j_1-i_1})\\
&\leqslant C_4\sum_{1\leqslant i_1\leqslant i_2<j_2\leqslant j_1\leqslant n}((j_2-i_2)^{-3/2}[1+O((j_2-i_2)^{-1})]\\
&\quad\times[(j_1-i_1-(j_2-i_2)+1]^{-3/2}[1+O((j_1-i_1-(j_2-i_2)+1)^{-1})]\\
&\quad-(j_1-i_1)^{-3/2}[1+O((j_1-i_1)^{-1})])\\
&\leqslant C_5\sum_{1\leqslant i_1\leqslant i_2<j_2\leqslant j_1\leqslant n}((j_2-i_2)^{-1/2}(j_1-i_1)^{-1}[j_1-i_1-(j_2-i_2)+1]^{-3/2}\\
&\quad+(j_2-i_2)^{-3/2}[(j_1-i_1-(j_2-i_2)+1)^{-5/2}+(j_1-i_1)^{-5/2}])\\
&\leqslant O(n\log n)
\end{aligned}$$

In fact, one can also get

$$L_1\geqslant O(n\log n)$$

We now consider $L_2$. Assume $i_1 \leqslant i_2 \leqslant j_1 < j_2$. By property (a) we can show that

$$P_n(0, x) \leqslant \bar{P}_n(0)[1 + O(n^{-1})]$$

Thus, if $j_2 - j_1 \geqslant i_2 - i_1$ and $p_{j_2 - j_1} > 0$,

$$\begin{aligned} &P_0(X(i_1) = X(j_1), X(i_2) = X(j_2)) \\ &\quad = P_0(X(j_1 - i_1) = 0, X(i_2 - i_1) = X(j_2 - i_1)) \\ &\quad = E_0(I_{\{X(j_1 - i_1) = 0\}} P_{j_2 - j_1}(X(j_1 - i_1), X(i_2 - i_1))) \\ &\quad \leqslant \bar{P}_{j_2 - j_1}(0)\, p_{j_1 - i_1}[1 + O((j_2 - j_1)^{-1})] \end{aligned}$$

and, if $j_2 - j_1 < i_2 - i_1$ and $p_{i_2 - i_1} > 0$,

$$\begin{aligned} &P_0(X(i_1) = X(j_1), X(i_2) = X(j_2)) \\ &\quad = P_0(X(j_2 - i_2) = 0, X(j_2 - i_1) = X(j_2 - j_1)) \\ &\quad \leqslant \bar{P}_{i_2 - i_1}(0)\, p_{j_2 - i_2}[1 + O((i_2 - i_1 + 1)^{-1})] \end{aligned}$$

Therefore, we have for some constants $C_6$, $C_7 \in (0, \infty)$

$$\begin{aligned} L_2 \leqslant & \sum_{1 \leqslant i_1 \leqslant i_2 \leqslant j_1 < j_2 \leqslant n} C_6 \{ I_{\{j_2 - j_1 \geqslant i_2 - i_1\}} \\ & \times (j_1 - i_1 + 1)^{-3/2}[(j_2 - j_1)^{-3/2} - (j_2 - i_2)^{-3/2} + O((j_2 - j_1)^{-5/2})] \\ & + I_{\{j_2 - j_1 < i_2 - i_1\}}(j_2 - i_2)^{-3/2}[(i_2 - i_1 + 1)^{-3/2} \\ & - (j_1 - i_1)^{-3/2} + O((j_2 - i_2 + 1)^{-5/2}))]\} \\ \leqslant & \, C_7 \sum_{i_1 = 0}^{n} \sum_{i_2 = 0}^{n - i_1} \sum_{j_1 = 0}^{n - i_1 - i_2} \sum_{j_2 = 1}^{n - i_1 - i_2 - j_1} \{ I_{\{i_2 \leqslant j_2\}} \\ & \times (j_1 + i_2 + 1)^{-3/2}[j_2^{-3/2}(j_2 + j_1)^{-1} j_1 + j_2^{-5/2}] \\ & + I_{\{i_2 > j_2\}}(j_2 + j_1 + 1)^{-3/2}[(i_2 + 1)^{-3/2}(j_1 + i_2)^{-1}(j_1 + 1) \\ & + (i_2 + 1)^{-5/2}]\} \\ \leqslant & \, O(n \log n) \end{aligned}$$

Combining the above estimates, we can get the desired result.

(ii) Without loss of generality, we may prove the desired result only for $d=4$. Let

$$A_1 = \left( \operatorname{Var} \left( I(nm) - \sum_{i=0}^{m-1} J_i(n) \right) \right)^{1/2}$$

$$A_2 = \left( \operatorname{Var} \left( \sum_{i=1}^{nm+k} \sum_{j=nm+1}^{(nm) \vee i} \delta(X_i, X_j) \right) \right)^{1/2}$$

where $J_i(n)$ was defined in the proof of Lemma 2.2. Then, we have for any given $n, m \geqslant 1$ and $k \leqslant n-1$

$$\operatorname{Var}(I(nm+k))^{1/2} \leqslant m^{1/2} \operatorname{Var}(I(n))^{1/2} + A_1 + A_2$$

As in the proof of Lemma 2.1, we can show that

$$A_2 \leqslant \operatorname{Var}(I(k))^{1/2} + O(\log(nm))$$

By a similar argument as in (i) we can show that

$$\operatorname{Var}(I(n)) = O(n)$$

Thus, we have for $k \leqslant n-1$

$$A_2 \leqslant O(I(n)) + O(\log(nm))$$

We now consider $A_1$. For simplicity, we only consider a special case: $n = 2^{n_1}$, $m = 2^{m_1}$. In this case,

$$A_1 = J(n_1, n_1 + m_1)$$

where $J(\cdot, \cdot)$ was defined in the proof of Theorem 2.3. By induction we have

$$\begin{aligned}
[E_0 J^2(n_1, n_1+m_1)]^{1/2} &\leqslant [E_0 J^2(n_1 - 1, n_1 + m_1)]^{1/2} \\
&\quad + \left( \sum_{k=1}^{2^{n_1}-1} \operatorname{Var} \left( \sum_{(i,j) \in T(n_1, k, n_1+m_1)} \delta(X_i, X_j) \right) \right)^{1/2} \\
&\leqslant [E_0 J^2(n_1 - 1, n_1 + m_1)]^{1/2} + O(2^{(n_1-1)/2} m_1) \\
&\leqslant O(m_1 2^{n_1/2})
\end{aligned}$$

In other words, we have

$$A_1 \leqslant O(n^{1/2} \log m)$$

Therefore, we have for $n, m \geqslant 1$ and $k \leqslant n-1$

$$\mathrm{Var}(I(nm+k))^{1/2} \leqslant m^{1/2}\,\mathrm{Var}(I(n))^{1/2} + O(n^{1/2}\log m) + O(\log(nm))$$

which implies that

$$\frac{\mathrm{Var}(I(nm+k))^{1/2}}{(nm)^{1/2}} \leqslant \frac{\mathrm{Var}(I(n))^{1/2}}{n^{1/2}} + O\left(\frac{\log m}{m^{1/2}}\right) + O\left(\frac{\log(nm)}{(nm)^{1/2}}\right)$$

Using this, we can show that

$$\limsup_{m\to\infty} \frac{\mathrm{Var}(I(m))^{1/2}}{m^{1/2}} \leqslant \liminf_{n\to\infty} \frac{\mathrm{Var}(I(n))^{1/2}}{n^{1/2}}$$

Hence, there is a constant $k_4 \in (0, \infty)$ such that

$$\lim_{n\to\infty} \frac{\mathrm{Var}(I(n))}{n^{1/2}} = k_4$$

which proves the desired result.

**Lemma 3.2.** Let us write $X_n = (X_n^1, ..., X_n^d) \in Z^d$, $\forall n \geqslant 0$. Then the following holds for any $k = 1, ..., d$:

$$\frac{E_0(X_n^k \cdot (I(n) - E_0 I(n)))}{n^{1/2}\alpha^{1/2}(n)\,\beta(n)} = 0$$

*Proof.* Note that $E_0 X_m^k = 0$, and that for $1 \leqslant i < j \leqslant n$

$$\begin{aligned}
&E_0(\delta(X_i, X_j)\, I_{\{X_n = x\}}) \\
&\quad = \sum_{y\in Z^d} P_0(X_i = y)\, P_0(X_{j-i} = 0, X_{n-i} = x - y) \\
&\quad = \sum_{y\in Z^d} P_0(X_i = y)\, P_0(X_{j-i} = 0)\, P_0(X_{n-j} = x - y) \\
&\quad = p_{j-i} \cdot P_0(X_{n-j+i} = x)
\end{aligned}$$

Then, we have

$$\begin{aligned}
&E_0(X_n^k(I(n) - E_0 I(n))) \\
&\quad = \sum_{i=1}^{n} \sum_{j=i+1}^{n} p_{j-i} E_0 X_{n-j+i}^k - E_0 X_n^k E_0 I(n) \\
&\quad = 0
\end{aligned}$$

which proves the desired result.

**Remark.** Here we have used the symmetry of the simple random walk $\{X_n\}$. We do not know how to extend the conclusion of Lemma 3.2 to a general random walk.

For any given $n \geq 1$, we let $\{(\eta_1(k, n),..., \eta_{d+1}(k, n)\}_{1 \leq k \leq n}$ be independent, identically distributed random vectors on the probability space $(\Omega, \mathscr{F}, P)$, and assume $\lim_{n \to \infty} \sigma_i(n) = \infty$, where

$$\sigma_i^2(n) = \text{Var}\left(\sum_{k=1}^{n} \eta_i(k, n)\right), \qquad i = 1,..., d+1$$

**Note.** Here we do not assume that the random variables $\eta_1(k, n),..., \eta_{d+1}(k, n)$ are independent for any given $k \leq n$ and $n \geq 1$.

**Lemma 3.3.** Assume that for $i = 1,..., d+1$

$$\lim_{n \to \infty} \sum_{k=1}^{n} (\eta_i(k, n) - E\eta_i(k, n))^4/\sigma_i^4(n) = 0 \tag{3.1}$$

It the following holds for any $1 \leq i < j \leq d$

$$\lim_{n \to \infty} \frac{E(\sum_{k=1}^{n} \eta_i(k, n) \cdot \sum_{k=1}^{n} \eta_j(k, n))}{\sigma_i(n) \cdot \sigma_j(n)} = 0 \tag{3.2}$$

then we have

$$\left(\frac{\sum_{k=1}^{n} (\eta_1(k, n) - E\eta_1(k, n))}{\sigma_1(n)},..., \frac{\sum_{k=1}^{n} (\eta_{d+1}(k, n) - \eta_{d+1}(k, n))}{\sigma_{d+1}(n)}\right)$$
$$\xrightarrow{(d)} N_{d+1}(0, 1)$$

as $n \to \infty$, where $N_{d+1}(0, 1)$ is a random vector in $R^{d+1}$ with the standard normal distribution.

*Proof.* For any given $t_1,..., t_{d+1} \in R^1$, we let

$$\eta(k, n) = \sum_{i=1}^{d+1} t_i \eta_i(k, n)/\sigma_i(n)$$

By (3.1) we know that the Lyapunov condition[2] is satisfied, i.e.,

$$\lim_{n \to \infty} E \sum_{k=1}^{n} |\eta(k, n) - E\eta(k, n)|^4 = 0$$

which implies that the Lindeberg condition is satisfied.[2] By (3.2) we can show that

$$\lim_{n\to\infty} E\left(\sum_{k=1}^{n} (\eta(k,n) - E\eta(k,n))\right)^2 = \sum_{i=1}^{d+1} t_i^2$$

Thus, we have

$$\frac{\sum_{k=1}^{n} (\eta(k,n) - E\eta(k,n))}{\sum_{i=1}^{d+1} t_i^2} \xrightarrow{(d)} N(0,1), \qquad n\to\infty$$

by the Lindeberg condition for the central limit theorem.[2] Let

$$\phi_n(t_1,\dots,t_{d+1}) = E\exp\left(i\sum_{l=1}^{d+1} t_l[\sigma_l(n)]^{-1}\sum_{k=1}^{n}[\eta_l(k,n) - E\eta_l(k,n)]\right)$$

Then

$$\phi_n(t_1,\dots,t_{d+1}) = E\exp\left(i\sum_{k=1}^{n}[\eta(k,n) - E\eta(k,n)]\right)$$
$$\to \exp(-\tfrac{1}{2}(t_1^2 + \cdots + t_{d+1}^2)), \qquad n\to\infty$$

In other words, $\phi_n$ converges to the characteristic function of $N_{d+1}(0, 1)$ as $n\to\infty$. From this we get the desired result. ∎

To state the next theorem, we first introduce some notations. For each $n\geqslant 1$, we construct a process $X^{(n)}\in C_0([0,\infty)\to R^d)$ as follows. Let

$$X^{(n)}(2^{-n}i) = 2^{-n/2}X_i, \qquad i = 0, 1, 2,\dots$$

and $X^{(n)}$ be linear on $[(i-1)2^{-n}, i2^{-n}]$ for $i = 1, 2,\dots$. Let

$$I(t,n) = \sum_{i=1}^{[tn]}\sum_{j=i+1}^{[tn]} \delta(X_i, X_j)$$

**Theorem 3.4.** (i) For $d = 3$, there are a sequence $\{u(n)\}$ and a constant $M\in[1,\infty)$ such that

$$M^{-1}\leqslant u(n)\leqslant M, \qquad \forall n\geqslant 1$$

and

$$\left(X^{(n)}(t), \frac{I(t,n) - E_0 I(t,n)}{n^{1/2}(\log n)^{1/2}\, u(n)}, t\geqslant 0\right)$$
$$\xrightarrow{(d)} (B_t, \beta_t, t\geqslant 0), \qquad n\to\infty \tag{3.3}$$

where $(B_t, \beta_t, t\geqslant 0)$ is the standard Brownian motion in $R^4$.

(ii) For $d \geqslant 4$, there is a constant $k_d \in (0, \infty)$ such that

$$\left(X^{(n)}(t), \frac{I(t, n) - E_0 I(t, n)}{n^{1/2} k_d}, t \geqslant 0\right)$$

$$\xrightarrow{(d)} (B_t, \beta_t, t \geqslant 0), \qquad n \to \infty$$

where $(B_t, \beta_t, t \geqslant 0)$ is the standard Brownian motion in $R^{d+1}$.

**Note.** In fact one can show that the sequence $\{u(n)\}$ in Theorem 3.4(i) can be chosen to be $u(n) = 2^{1/2}(2\pi)^{-1}$, $\forall n \geqslant 1$.

*Proof.* (i) By Lemma 3.1 and a sample calculation we can show that

$$\beta(n) = O(n^{1/2}), \qquad \alpha(n) = O(\log n)$$

Let $\tilde{\alpha}(n) = \log n$, $\tilde{\beta}(n) = n^{1/2}$, and

$$u(n) = \tilde{\alpha}(n)^{-1/2} \tilde{\beta}(n)^{-1} \alpha^{1/2}(n)\, \beta(n)$$

Then we have

$$u(n) = O(1), \qquad n \geqslant 1$$

It is easy to show that the simple random walk in $Z^3$ satisfies all assumptions given in Lemma 2.2. For simplicity, we only prove that the desired result is true if the sequence $\{n\}$ is replaced by the sequence $\{2^n, n \geqslant 1\}$. In order words, we prove that

$$\left(X^{(2^n)}(t), \frac{\tilde{I}(t, 2^n)}{2^{n/2}(\log 2^n)^{1/2}\, u(2^n)}, t \geqslant 0\right)$$

$$\xrightarrow{(d)} (B_t, \beta_t, t \geqslant 0), \qquad n \to \infty \tag{3.4}$$

where

$$\tilde{I}(t, 2^n) = I(t, 2^n) - E_0 I(t, 2^n)$$

In fact, the idea to prove (3.3) is the same as that to prove (3.4), it is just that for (3.3) more notations have to be introduced.

First, we prove that for any fixed $t > 0$

$$\left(X^{(2^n)}(t), \frac{\tilde{I}(t, 2^n)}{2^{n/2}(\log 2^n)^{1/2}\, u(2^n)}\right) \xrightarrow{(d)} (B_t, \beta_t), \qquad n \to \infty \tag{3.5}$$

Let $m_n = \log_2 n^{1/4}$, $\forall n \geqslant 1$. Then (2.4) holds with this choice. Let

$$\tilde{J}(t, n) = \sum_{i=0}^{[t2^{m_n}]-1} [J_i(2^{n-m_n}) - E_0 J_i(2^{n-m_n})]$$

From the proof of Theorem 2.3 we see that (3.5) follows from

$$\left(\frac{\sum_{i=0}^{[t2^{m_n}]-1} \xi_i(n)}{2^{m_n/2}}, \frac{\tilde{J}(t, n)}{\beta(2^{n-m_n})\, 2^{m_n/2} \alpha^{1/2}(2^{n-m_n})}\right) \xrightarrow{(d)} (B_t, \beta_t), \qquad t \geqslant 0 \qquad (3.6)$$

where

$$\xi_i(n) = \sum_{k=i2^{nm_n}+1}^{(i+1)2^{n-m_n}} 2^{-(n-m_n)/2} \xi_k =: (\xi_i^1(n), \xi_i^2(n), \xi_i^3(n))$$

We now use Lemma 3.3 to prove (3.6). Let

$$\sigma_l^2(t, n) = \operatorname{Var}\left(\sum_{i=0}^{[t2^{m_n}]-1} \xi_i^l(n)\right), \qquad l = 1, 2, 3$$

Then

$$\sigma_l^2(t, n) = t2^{m_n}[1 + o(1)]$$

It is easy to show that

$$\sum_{i=0}^{[t2^{m_n}]-1} E_0 |\xi_i(n) - E_0 \xi_i(n)|^4 \leqslant O(t2^{m_n})$$

Hence

$$\lim_{n \to \infty} \sum_{i=0}^{[t2^{m_n}]-1} \sigma_i^{-4}(t, n) E_0 |\xi_i(n) - E_0 \xi_i(n)|^4 = 0, \qquad l = 1, 2, 3$$

Moreover, we set

$$\sigma_4^2(t, n) = \operatorname{Var}\left(\sum_{i=0}^{[t2^{m_n}]-1} J_i(2^{n-m_n})\right)$$

As in the derivation of (2.8), we can show that [the following estimate was shown in (2.8) to be true for $t = 1$]

$$\begin{aligned} &\sum_{i=0}^{[t2^{m_n}]-1} E_0(J_i(2^{n-m_n}) - E_0 J_i(2^{n-m_n}))^4 \\ &\leqslant t2^{m_n} \beta^2(2^{n-m_n})\, \alpha(2^{n-m_n}) \\ &\leqslant t2^{-m_n} \sigma_4^4(t, n) \end{aligned}$$

Using this, we can easily show that

$$\lim_{n\to\infty} \sum_{i=0}^{[t2^{m_n}]-1} \frac{E_0(J_i(2^{n-m_n}) - E_0 J_i(2^{n-m_n}))^4}{\sigma_4^4(t, n)} = 0$$

Thus we proved that the conditin (3.1) is satisfied.

As in the proof of (2.5), we can show that

$$E_0 \, |X^l_{[t2^n]}(\tilde{I}(t, 2^n) - \tilde{J}(t, n))| \leqslant O(m_n \cdot 2^{n/2})(E_0 \, |X^l_{[t2^n]}|^2)^{1/2}$$

Then by Lemma 3.2 we know that

$$\lim_{n\to\infty} \left(2^{-n} n^{-1/2} E_0 \left(\sum_{i=0}^{[t2^{m_n}]-1} \xi_i^l(n) \, \tilde{J}(t, n)\right)\right)$$

$$= \lim_{n\to\infty} (2^{-n} n^{-1/2} E_0 X^l_{[t2^n]} \cdot (I(t, 2^n) - E_0 I(t, 2^n))) = 0$$

which proves that the condition (3.2) is satisfied. Therefore, by Lemma 3.3 we have

$$\left(\frac{\sum_{i=0}^{[t2^{m_n}]-1} \xi_i(n)}{t^{1/2} 2^{m_n/2}}, \frac{\tilde{J}(t, n)}{t^{1/2} \beta(2^{n-m_n}) \, 2^{m_n/2} \, \alpha^{1/2}(2^{n-m_n})}\right)$$

$$\xrightarrow{(d)} N_{3+1}(0, 1), \qquad n \to \infty$$

which proves (3.6), and so the proof of (3.5) is complete. Similarly, we can show that for $t_1 < t_2$

$$\left(X^{(2^n)}(t_2) - X^{(2^n)}(t_1), \frac{\tilde{J}(t_2, n) - \tilde{J}(t_1, n)}{\beta(2^{n-m_n}) \, 2^{m_n/2} \, \alpha^{1/2}(2^{n-m_n})}\right)$$

$$\xrightarrow{(d)} (B_{t_2-t_1}, \beta_{t_2-t_1}), \qquad n \to \infty \tag{3.7}$$

Finally, we prove (3.4). In fact, we only need to show that for any $0 = t_0 < t_1 < t_2 < \cdots < t_k < \infty$, $\forall k \geqslant 1$,

$$\left(X^{(2^n)}(t_l), \frac{\tilde{I}(t_l, 2^n)}{2^{n/2}(\log 2^n))^{1/2} \, u(2^n)}, 1 \leqslant l \leqslant k\right)$$

$$\xrightarrow{(d)} (B_{t_l}, \beta_{t_l}, 1 \leqslant l \leqslant k), \qquad n \to \infty$$

To prove this, it suffices to prove that

$$\left(X^{(2^n)}(t_l)-X^{(2^n)}(t_{l-1}), \frac{\tilde{I}(t_l, 2^n)-\tilde{I}(t_{l-1}, 2^n)}{2^{n/2}(\log 2^n)^{1/2}\, u(2^n)}, 1 \leqslant l \leqslant k\right)$$

$$\xrightarrow{(d)} (B_{t_l}-B_{t_{l-1}}, \beta_{t_l}-\beta_{t_{l-1}}, 1 \leqslant l \leqslant k), \qquad n \to \infty$$

As in the proof of (2.5), we can show that

$$E_0 \left| \frac{\tilde{I}(t_l, 2^n)-\tilde{I}(t_{l-1}, 2^n)}{2^{n/2}(\log 2^n)^{1/2}\, u(2^n)} - \frac{\tilde{J}(t_l, n)-\tilde{J}(t_{l-1}, n)}{2^{n/2}(\log 2^n)^{1/2}\, u(2^n)} \right| \to 0, \qquad n \to \infty$$

Thus, it suffices to show that

$$\left(X^{(2^n)}(t_l)-X^{(2^n)}(t_{l-1}), \frac{\tilde{J}(t_l, n)-\tilde{J}(t_{l-1}, n)}{2^{n/2}(\log 2^n)^{1/2}\, u(2^n)}, 1 \leqslant l \leqslant k\right)$$

$$\xrightarrow{(d)} (B_{t_l}-B_{t_{l-1}}, \beta_{t_l}-\beta_{t_{l-1}}, 1 \leqslant l \leqslant k), \qquad n \to \infty \tag{3.8}$$

It is clear that the random vectors

$$(X^{(2^n)}(t_l)-X^{(2^n)}(t_{l-1}), \tilde{J}(t_l, n)-\tilde{J}(t_{l-1}, n), 1 \leqslant l \leqslant k)$$

are independent. Hence, (3.8) is actually an immediate consequence of (3.7). Since the tightness can be easily proved to be true in this case, we get the weak convergence (3.3) from (3.8).

(ii) Here we choose $m_n = n^{1/2}$, $\forall n \geqslant 1$,

$$\tilde{\alpha}(n) = \begin{cases} n/\log^2 n, & d=4 \\ n, & d \geqslant 5 \end{cases}$$

and

$$\tilde{\beta}(n) = \begin{cases} \log n, & d=4 \\ 1, & d \geqslant 5 \end{cases}$$

We define $u(n)$ as in the proof of (i). Then, by Lemma 3.1(ii) we can show that

$$\lim_{n \to \infty} u(n) = k_d$$

for some constant $k_d \in (0, \infty)$. Thus, by a similar argument as in the proof of (i), part (ii) follows.

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